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NEW STEADY AND SELF-SIMILAR SOLUTIONS OF THE EULER EQUATIONS

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Exact steady and self-similar solutions of the Euler equations are considered, which possess the property of partial invariance with respect to a certain six-parameter Lie group. New examples of vortex motion of a swirled liquid in curved channels are presented. A classification is given for self-similar solutions of the reduced system with two independent variables, which admits a three-parameter group of extensions, whereas the initial system of the Euler equations possesses a two-parameter group.

Key words: *Euler equations, partially invariant solutions, streamlines, sources, drains.*

1. Group-Theoretical Nature of the Solution. We consider a partially invariant solution of the Euler equations that describe rotationally symmetric motion of an ideal incompressible liquid [1]. For constructing the partially invariant solution, the system of the Euler equations is written in cylindrical coordinates r, θ, z. The projections of the velocity vector $\boldsymbol{v}$ onto the corresponding axes are denoted by u, v, and w. According to the universal algorithm of constructing partially invariant solutions [2], the vertical component of velocity w is a function of two variables: the vertical coordinate z and the time t, whereas two other components of velocity u and v and the pressure p are independent of the polar angle θ:

$$w = w(z,t), \qquad u = u(r,z,t), \qquad v = v(r,z,t), \qquad p = p(r,z,t).$$

Substituting this representation of solutions into the Euler equations written in cylindrical coordinates, we obtain the system

$$\begin{gathered} u_t + uu_r + wu_z - r^{-1}v^2 + p_r = 0, \qquad v_t + uv_r + wv_z + r^{-1}uv = 0, \\ w_t + ww_z + p_z = 0, \qquad u_r + r^{-1}u + w_z = 0, \end{gathered} \tag{1.1}$$

which was considered in [3, 4]. From the last equation of (1.1), by integration ($w_r = 0$), we can easily obtain the function

$$u = -rw_z/2 + q/r, \tag{1.2}$$

where $q(z,t)$ is a new unknown function. From the resultant overdetermined system, we can also find the velocity component v [1]

$$v = r^{-1}[r^4(a+s) + r^2 b - q^2]^{1/2},$$

where $a = -w_{zt}/2 - ww_{zz}/2 + w_z^2/4$, $b = q_t + wq_z$, and the function $s(r,t)$ is determined by solving the equation

$$r^2(a_t + wa_z - 2w_z a) + r^2 s_t - r^2 w_z(rs_r + 4s)/2 + b_t + wb_z - w_z b + 4qa + q(rs_r + 4s) = 0.$$

In analyzing (1.1), one should consider several cases, each leading to a certain class of solutions that describe swirling motion. For $w_{zz} \neq 0$, the overdetermined system takes the form

$$\begin{gathered} f_t + wf_z + f^2 - a = 0, \qquad a_t + wa_z + 4(a+\chi)f + \dot\chi = 0, \\ q_t + wq_z - b = 0, \qquad b_t + wb_z + 2fb + 4(a+\chi)q = 0, \end{gathered} \tag{1.3}$$

where $f = -w_z/2$.

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Writing system (1.3) in Lagrangian coordinates, we can note that it admits exact linearization. The class of solutions for this particular unsteady case and examples of possible vortex flows are described in [3, 4]. Steady and self-similar solutions corresponding to system (1.3) are considered in the present paper.

2. Steady Solutions. We consider steady solutions of system (1.3) assuming that $w = w(z)$, $a = a(z)$, $q = q(z)$, $b = b(z)$, and $\chi = \text{const}$. As a result, we obtain a system of equations with respect to w and q:

$$2ww'' - w'^2 + 4a = 0, \qquad wa' - 2(a+\chi)w' = 0, \qquad wq' - b = 0, \qquad wb' - w'b + 4(a+\chi)q = 0. \tag{2.1}$$

Note, the variables are separated in the second equation, and integration yields

$$a = Cw^2/4 - \chi, \tag{2.2}$$

where C is an arbitrary constant of integration.

Substituting the result obtained into the first equation of system (2.1), we obtain the second-order nonlinear differential equation

$$2ww'' - w'^2 + Cw^2 - 4\chi = 0. \tag{2.3}$$

Let us reduce the order of this equation by assuming that $w' = g(w)$. Then, by virtue of $w'' = w'\dot{g} = g\dot{g}$, Eq. (2.3) takes the form

$$2wg\dot{g} - g^2 + Cw^2 - 4\chi = 0.$$

Assuming further that $g^2 = h$, we rewrite this equation as

$$w\dot{h} - h - 4\chi + Cw^2 = 0.$$

This equation has the general solution

$$h = C_1 w - Cw^2 - 4\chi$$

or (with allowance for the definition of h)

$$w'^2 = C_1 w - Cw^2 - 4\chi. \tag{2.4}$$

Solving Eq. (2.4), we obtain

$$w = \begin{cases} (C_1 \pm \sqrt{C_1^2 - 16\chi C}\,\sin(\sqrt{C}\,z - C_2\sqrt{C}\,))/(2C), & C > 0,\ C_1^2 - 16\chi C \geqslant 0, \\ (C_1 \pm \sqrt{16\chi C - C_1^2}\,\sinh(\sqrt{|C|}\,z - C_2\sqrt{|C|}\,))/(2C), & C < 0,\ C_1^2 - 16\chi C < 0. \end{cases} \tag{2.5}$$

By virtue of the third equation of (2.1) and expression (2.2), the fourth equation of (2.1) reduces to the simple equation

$$q'' + Cq = 0,$$

whose solution has the form

$$q = \begin{cases} A_1 \sin(\sqrt{C}\,z) + B_1 \cos(\sqrt{C}\,z), & C > 0, \\ A_2 \cosh(\sqrt{|C|}\,z) + B_2 \sinh(\sqrt{|C|}\,z), & C < 0. \end{cases} \tag{2.6}$$

We find the streamlines on the basis of the definition $dr/u = r\,d\theta/v = dz/w$ or, in our case, $dr/u = dz/w$, which yields $w\,dr - u\,dz = 0$. With allowance for (1.2), we obtain $r^2 w_z\,dz + 2rw\,dr - 2q\,dz = 0$ or $d(r^2 w) - 2\,d\Big(\int\limits_0^z q\,dz'\Big) = 0$. Then, we have the equation for the surface formed by the streamlines:

$$r^2 w - 2\int\limits_0^z q\,dz' = \text{const}. \tag{2.7}$$

We denote the left side of Eq. (2.7) as $\psi(r,z)$. Substituting (2.5) and (2.6) into (2.7), we find the dependence of r on z:

$$r^2(C_1 \pm \sqrt{C_1^2 - 16\chi C}\,\sin(\sqrt{C}\,z - C_2\sqrt{C}\,))/(2C) + A_1\cos(z\sqrt{C}\,)/\sqrt{C} - B_1\sin(z\sqrt{C}\,)/\sqrt{C} = \text{const}; \tag{2.8a}$$

$$\begin{aligned} &r^2(C_1 \pm \sqrt{16\chi C - C_1^2}\,\sinh(\sqrt{|C|}z - C_2\sqrt{|C|}\,))/(2C) \\ &- 2A_2\sinh(z\sqrt{|C|})/\sqrt{|C|} - 2B_2\cosh(z\sqrt{|C|}\,)/\sqrt{|C|} = \text{const}. \end{aligned} \tag{2.8b}$$

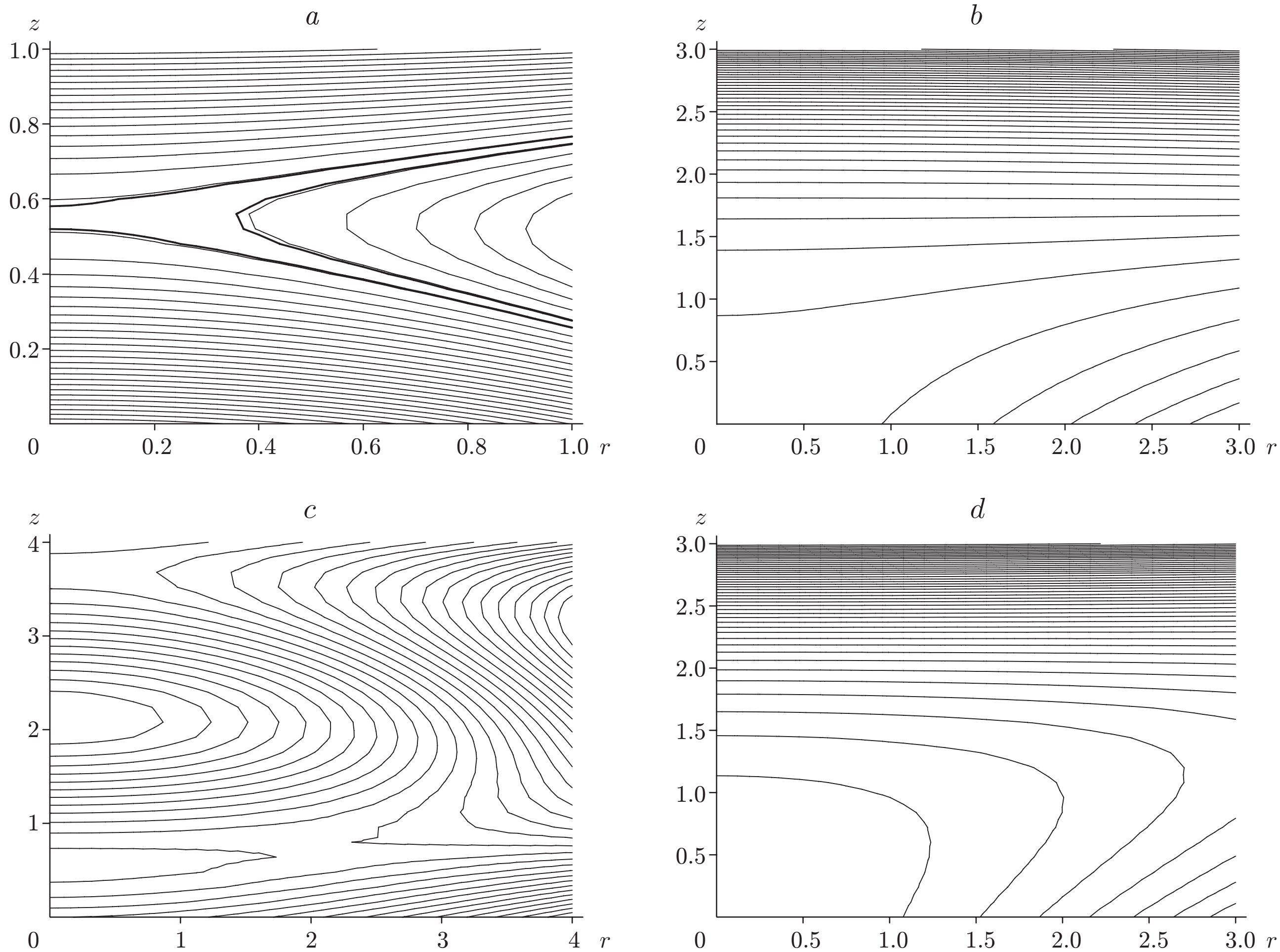

Fig. 1. Streamlines calculated by formulas (2.8a) with the minus sign (a), (2.8b) with the plus sign (b), (2.8a) with the plus sign (c), and (2.8b) with the minus sign (d) for $C = 1$, $C_1 = 1$, $C_2 = 1$, $A_1 = 5$, $B_1 = -10$, and $\chi = -1$ (a and c) and $C = -1$, $C_1 = 1$, $C_2 = 2$, $A_2 = 1$, $B_2 = 1$, and $\chi = -1$ (b and d).

Formula (2.8a) corresponds to the case $C > 0$ and $C_1^2 - 16\chi C \geqslant 0$; formula (2.8b) corresponds to the case $C < 0$ and $C_1^2 - 16\chi C < 0$. Hereinafter, we refer to Eq. (2.8a) if we use the constants A_1 and B_1 and to Eq. (2.8b) if we use the constants A_2 and B_2.

The lines $\psi(r, z) = \text{const}$ are streamlines in the meridional cross section. Following [5, 6], the set of levels of stream functions will be called the flow portrait. By virtue of the large parametric arbitrariness (C, C_i, A_i, B_i, and χ; $i = 1, 2$), there is a rather large class of flow portraits represented by various channels with branching. Figure 1 shows some typical flow portraits; the cases with the plus or minus sign chosen in Eqs. (2.8a) and (2.8b) are distinguished.

We determine the flow rate of the liquid through the cross section $r = 0.6$ for a chosen curved channel (Fig. 1a). The inner and outer walls of the channel are specified by the equations $\psi(r, z) = 11.355$ and $\psi(r, z) = 11.155$, respectively, where the function ψ is determined by formulas (2.8). In our case, the flow rate equals the product of π and the difference in the stream-function values. Solving the equations $\psi(0.6, z) = 11.355$ and $\psi(0.6, z) = 11.155$, we find $z_1 = 0.660097$ and $z_2 = 0.692672$ for the upper branch of the channel and $z_1 = 0.384397$ and $z_2 = 0.417106$ for the lower branch. We denote the flow rate through the upper and lower branches of the channel by Q_1 and Q_2, respectively. Finally, we obtain $Q_1 = -0.628\,319$ and $Q_2 = 0.628\,319$, i.e., the liquid comes in through the upper channel and goes out through the lower channel, providing zero flow rate through a given cross section as a whole.

It is known that rotationally symmetric steady motion (swirling motions) of an ideal liquid can be described by the Grad–Shafranov equation [5–7]

$$\psi_{zz} + \psi_{rr} - \psi_r/r = r^2 G(\psi) + F(\psi), \tag{2.9}$$

where G and F are arbitrary functions of ψ. The cases of integrability of Eq. (2.9) and the characteristic flow portraits can be found in [5–7]. It can be shown that the solution constructed in the present paper does not enter any known class. Because of the explicit form of the stream function (2.8), we can introduce the notation $\psi = r^2 a(z) + b(z)$; substituting the latter into Eq. (2.9), we obtain

$$r^2 a_{zz} + b_{zz} = r^2 G(\psi) + F(\psi). \tag{2.10}$$

Denoting $s = r^2$ and differentiating (2.10) in terms of s, we obtain the equation

$$G(\psi) + aF_\psi(\psi) + asG_\psi(\psi) = a_{zz}.$$

Substituting the expression for s, we obtain

$$G(\psi) + aF_\psi(\psi) + aG_\psi(\psi)(b_{zz} - F(\psi))/(G(\psi) - a_{zz}) = a_{zz}. \tag{2.11}$$

Differentiating Eq. (2.11) with respect to ψ, we find

$$2GG_\psi + a(F_\psi G - FG_\psi)_\psi - 2a_{zz}G_\psi - aa_{zz}F_{\psi\psi} + ab_{zz}G_{\psi\psi} = 0.$$

Then, we differentiate with respect to z:

$$a_z(F_\psi G - FG_\psi)_\psi - 2a_{zzz}G_\psi - (aa_{zz})_z F_{\psi\psi} + (ab_{zz})_z G_{\psi\psi} = 0.$$

Dividing by G_ψ (assuming that $G_\psi \neq 0$) and differentiating with respect to ψ, we obtain

$$a_z((F_\psi G - FG_\psi)/G_\psi)_\psi - (aa_{zz})_z(F_{\psi\psi}/G_\psi)_\psi + (ab_{zz})_z(G_{\psi\psi}/G_\psi)_\psi = 0.$$

We divide this equation by a_z [note, a_z is not identically equal to zero, but we should assume that $z \neq (2C_2\sqrt{C} + \pi)/(2\sqrt{C}\,)$] and differentiate with respect to z. We obtain the following equation [here we also assume that the quantity $[(aa_{zz})_z/a_z)_z$ is considered at points where it does not vanish]:

$$\Big(\frac{F_{\psi\psi}}{G_\psi}\Big)_\psi - \frac{((ab_{zz})_z/a_z)_z}{((aa_{zz})_z/a_z)_z}\Big(\frac{G_{\psi\psi}}{G_\psi}\Big)_\psi = 0.$$

After differentiation with respect to ψ, this equation reduces to simple differential equations

$$(G_{\psi\psi}/G_\psi)_{\psi\psi} = 0, \qquad (F_{\psi\psi}/G_\psi)_{\psi\psi} = 0,$$

which, nevertheless, have no solutions in elementary functions

$$G = \int \exp(k\psi^2 + l\psi + n)\,d\psi, \qquad F = \int\Big(\int (k_1\psi + l_1)\exp(k\psi^2 + l\psi + n)\,d\psi\Big)\,d\psi,$$

where k, k_1, l, l_1, and n are arbitrary constants of integration.

Thus, a new class of partially invariant steady solutions of the Euler equations, which describes swirling motions in curved channels, has been obtained.

3. Self-Similar Solutions. Since the group of extensions was not used in constructing the solutions from [1, 3, 4], we present a two-parameter group of extensions for (1.1):

$$X_1 = r\,\partial_r + z\,\partial_z + u\,\partial_u + v\,\partial_v + w\,\partial_w + 2p\,\partial_p, \quad X_2 = t\,\partial_t - u\,\partial_u - v\,\partial_v - w\,\partial_w - 2p\,\partial_p.$$

For the reduced system (1.3), together with the equation $f = -w_z/2$, there arises the problem of group classification in terms of the element $\chi(t)$, which is not considered here. We take the function χ in the form $\chi = c/t^n$, where c and n are constants. It should be noted that, for an arbitrary $n \neq 2$, system (1.3) admits only the trivial extension transformation $q \to kq$, $b \to kb$ ($k = \text{const}$). For $n = 2$, the group of extensions admitted by Eqs. (1.3) has three parameters, and its generators have the form

$$Y_1 = t\,\partial_t - w\,\partial_w - f\,\partial_f - 2a\,\partial_a + q\,\partial_q, \quad Y_2 = z\,\partial_z + w\,\partial_w, \quad Y_3 = b\,\partial_b + q\,\partial_q. \tag{3.1}$$

We consider three operators, which are a linear combination of operators (3.1): $Y_1 + \beta Y_2 + \gamma Y_3$, $Y_2 + \delta Y_3$, and Y_3. Note, it is impossible to construct an invariant solution with respect to the operator Y_3, since the necessary conditions of its existence are not satisfied. In the case of the operator $Y_2 + \delta Y_3$, the representation of solutions has the following form ($\delta = \text{const}$):

$$w = -2z\varphi(t), \qquad b = z^\delta\psi(t), \qquad q = z^\delta\omega(t).$$

In this case, denoting $a + ct^{-2} = \pm g^2$, we rewrite system (1.3) in the form

$$\varphi' + \varphi^2 - a = 0, \quad (\pm g^2)' + 4(\pm g^2)\varphi = 0, \quad \omega' - 2\delta\varphi\omega - \psi = 0, \quad \psi' - 2\delta\varphi\psi + 2\varphi\psi + 4(\pm g^2)\omega = 0. \tag{3.2}$$

As it follows from the form of system (3.2), the first two equations are split and integrated separately. For instance, by choosing the minus sign at g^2, we find

$$\varphi' + \varphi^2 + g^2 + c/t^2 = 0, \qquad g' + 2\varphi g = 0.$$

Denoting $\varphi + g = \lambda$ and $\varphi - g = \mu$, we obtain the Riccati equations for the functions λ and μ:

$$\lambda' + \lambda^2 + c/t^2 = 0, \qquad \mu' + \mu^2 + c/t^2 = 0.$$

Solving these equations and using the definitions of λ and μ, we obtain

$$\varphi = \frac{1}{4t}\Big[2 + \sqrt{4c-1}\Big(\tan\Big(-\frac{\sqrt{4c-1}}{2}\ln t + \frac{C_1\sqrt{4c-1}}{2}\Big) + \tan\Big(-\frac{\sqrt{4c-1}}{2}\ln t + \frac{C_2\sqrt{4c-1}}{2}\Big)\Big)\Big],$$

$$g = \frac{1}{4t}\Big[\sqrt{4c-1}\Big(\tan\Big(-\frac{\sqrt{4c-1}}{2}\ln t + \frac{C_1\sqrt{4c-1}}{2}\Big) - \tan\Big(-\frac{\sqrt{4c-1}}{2}\ln t + \frac{C_2\sqrt{4c-1}}{2}\Big)\Big)\Big],$$

where C_1 and C_2 are arbitrary constants of integration.

For $a + ct^{-2} = g^2$, the system for the functions φ and g reduces to the complex Riccati equation, which is exactly solved in elementary functions.

Despite their linearity, the last two equations in (3.2) have no simple solutions and, hence, are not considered here.

The widest class of self-similar solutions is obtained on the group with the operator $Y_1 + \beta Y_2 + \gamma Y_3$. We denote the independent self-similar variable as $\zeta = zt^{-\beta}$, then the sought functions have the form

$$w = \lambda(\zeta)/t^2, \quad a = \mu(\zeta)/t^2, \quad \chi = c/t^2, \quad b = \eta(\zeta)/t^\gamma, \quad q = \sigma(\zeta)/t^{\gamma-1}. \tag{3.3}$$

Choosing $\beta = -1$, i.e., the self-similar variable $\zeta = zt$, we substitute functions (3.3) into system (1.3):

$$\begin{gathered} 2(\lambda+\zeta)\lambda'' - 2\lambda' - \lambda'^2 + 4\mu = 0, \qquad (\lambda+\zeta)\mu' - 2(\lambda'+1)(\mu+c) = 0, \\ (\lambda+\zeta)\sigma' - (\gamma-1)\sigma = \eta, \qquad \eta'(\lambda+\zeta) - \eta(\lambda'+\gamma) + 4(\mu+c)\sigma = 0. \end{gathered} \tag{3.4}$$

The second equation of the system yields

$$K(\lambda+\zeta)^2 = \mu + c. \tag{3.5}$$

We denote $\nu \equiv \lambda + \zeta$. With allowance that $\nu'' = \lambda''$, the first equation of (3.4) is rewritten as follows:

$$2\nu\nu'' - \nu'^2 + 4\mu + 1 = 0.$$

By virtue of (3.5), we have $4\mu = 4K\nu^2 - 4c$. Then,

$$2\nu\nu'' - \nu'^2 + 4K\nu^2 + 1 - 4c = 0. \tag{3.6}$$

To solve this equation, we use the same algorithm involved in solving Eq. (2.3). After integration of Eq. (3.6), we obtain the equation

$$(\nu')^2 = K_1\nu - 4K\nu^2 + 1 - 4c,$$

which, in turn, has the following solution (by virtue of the definition of ν):

$$\lambda(\zeta) = -(8\zeta K - K_1 \pm \sqrt{K_1^2 - 64cK + 16K}\,\sin(2\sqrt{K}\,\zeta - 2K_2\sqrt{K}\,))/(8K); \tag{3.7a}$$

$$\lambda(\zeta) = -(8\zeta K - K_1 \pm \sqrt{64cK - K_1^2 - 16K}\,\sinh(2\sqrt{|K|}\,\zeta - 2K_2\sqrt{|K|}\,))/(8K). \tag{3.7b}$$

Formula (3.7a) is valid for $K > 0$ and $K_1^2 - 64cK + 16K \geqslant 0$; formula (3.7b) is valid for $K < 0$ and $K_1^2 - 64cK + 16K < 0$ (K, K_1, and K_2 are arbitrary constants of integration).

The last two equations in (3.4) are split from the first two equations, but they have a simple analytical solution for $\gamma = 1$ only. In this case, the equations are simplified:

$$(\lambda+\zeta)\sigma' = \eta, \qquad \eta'(\lambda+\zeta) - \eta(\lambda'+1) + 4(\mu+c)\sigma = 0.$$

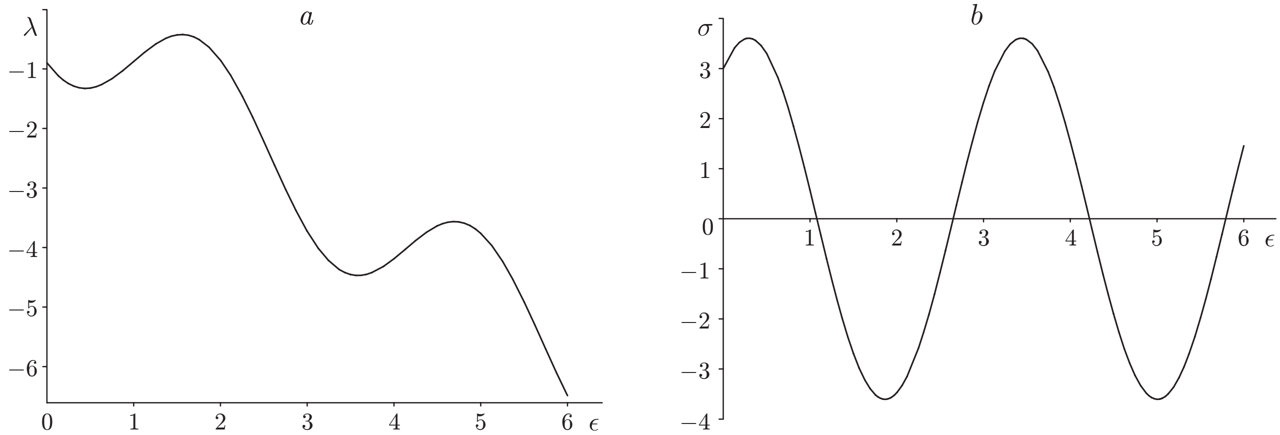

Fig. 2. Solutions of equations (3.7a) with the minus sign (a) and (3.8a) (b) ($K = 1$, $K_1 = 1$, $K_2 = 1$, $K_3 = 2$, $K_4 = 3$, and $c = -1$).

Substituting the expressions for η into the second equation and using (3.5), we obtain a simple equation with respect to σ

$$\sigma'' + 4K\sigma = 0,$$

whose solution has form similar to (2.6):

$$\sigma = K_3 \sin(2\sqrt{K}\,\zeta) + K_4 \cos(2\sqrt{K}\,\zeta), \qquad K > 0; \tag{3.8a}$$

$$\sigma = K_5 \cosh\,(2\sqrt{|K|}\,\zeta) + K_6 \sinh\,(2\sqrt{|K|}\,\zeta), \qquad K < 0. \tag{3.8b}$$

Figure 2 shows the dependences $\lambda(\zeta)$ and $\sigma(\zeta)$ for particular values of parameters.

It should be noted that, for $\gamma = 1$, the function q (density of sources or drains) coincides with the function σ. Therefore, the solution obtained can be considered as the motion induced by sources and drains distributed along the z axis with a period πt^{-1}. The spatial period tends to zero as $t \to \infty$, which means concentration of vortex sources at the axis of symmetry.

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